BAB IV
PEMBAHASAN

4.1 Sistem Dinamik Diskrit Untuk Matriks $2x2$

Dibentuk persamaan karakteristik dinamik diskrit :

$$y(k) + a_1 y(k-1) + a_2 y(k-2) = b_0 u(k) + b_1 u(k-1) + b_2 u(k-2) \tag{4.1}$$

Dengan transformasi z sebagai berikut:

$y(k) = y(z)$ dan $y(k-i) = z^{-i} y(k)$ dan $u(k) = u(z)$ dan $u(k-i) = z^{-i} u(z)$

Maka persamaan (4.1) menjadi :

$$y(z) + a_1 z^{-1} y(z) + a_2 z^{-2} y(z) = b_0 u(z) + b_1 z^{-1} u(z) + b_2 z^{-2} u(z)$$

$$y(z)(1 + a_1 z^{-1} + a_2 z^{-2}) = u_{(z)}(b_0 + b_1 z^{-1} + b_2 z^{-2})$$

Kemudian dibentuk ke persamaan *Pulse Transfer* sebagai berikut:

$$\frac{y(z)}{u(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

atau

$$\frac{y(z)}{u(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \tag{4.2}$$

Persamaan (4.2) direpresentasikan ke persamaan kanonik terkontrol waktu diskrit sebagai berikut:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k \tag{4.3}$$

Diberikan fungsi tujuan yaitu:

$$J = \frac{1}{2} \boldsymbol{x}_N^T S_N \boldsymbol{x}_N + \frac{1}{2} \sum_{k=1}^{N-1} (\boldsymbol{x}_k^T Q_k \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k), \tag{4.4}$$

Pada persamaan (4.4) diasumsikan Q_k, R_k, dan S_N adalah matrik simetri semi definit positif dan $R_k \neq 0$ untuk semua k.

Dari persamaan (4.3) dan (4.4) dibentuk persamaan Hamilton sebagai berikut:

$$\boldsymbol{H} = \frac{1}{2}\left(\boldsymbol{x}^T Q_k \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k\right) + \lambda_{k+1}^T \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k \right), \tag{4.5}$$

Kemudian persamaan (4.5) menghasilkan persamaan *state*, *costate*, dan *stationer*

Persamaan *state* : $\boldsymbol{x}_{k+1} = \frac{\partial H}{\partial \lambda_{k+1}} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k,$ (4.6)

Persamaan *costate* : $\lambda_k = \frac{\partial H}{\partial x_k} = Q_k \boldsymbol{x}_k + \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T \lambda_{k+1},$ (4.7)

Persamaan *stationer* : $0 = \frac{\partial H}{\partial u_k} = R_k \boldsymbol{u}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \lambda_{k+1},$ (4.8)

Menurut persamaan (4.8) diperoleh:

$$\boldsymbol{u}_k = -R_k^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \lambda_{k+1}, \tag{4.9}$$

Kemudian persamaan (4.9) disubstitusikan ke persamaan (4.6):

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 \\ 1 \end{bmatrix} R_k^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \lambda_{k+1}, \tag{4.10}$$

Menurut (**Ogata, 1995**) Selanjutnya diasumsikan untuk setiap k berlaku:

$$\lambda_k = S_k \boldsymbol{x}_k \tag{4.11}$$

Substitusikan persamaan (4.11) kedalam (4.10) untuk mendapatkan persamaan:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 \\ 1 \end{bmatrix} R_k^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T S_{k+1} \boldsymbol{x}_{k+1} \tag{4.12}$$

Pemecahan untuk $\boldsymbol{x}_{k+1}$ menghasilkan:

$$\boldsymbol{x}_{k+1} = \left(I + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R_k^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T S_{k+1} \right)^{-1} \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \boldsymbol{x}_k. \tag{4.13}$$

Selanjutnya, substitusikan persamaan (4.11) ke persamaan *costate* (4.7), menghasilkan:

$$S_k \boldsymbol{x}_k = Q_k \boldsymbol{x}_k + \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T S_{k+1} \boldsymbol{x}_{k+1}, \tag{4.14}$$

Kemudian, substitusikan persamaan (4.13) ke persamaan (4.14), didapat:

$$S_k = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T S_{k+1} \left(I + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R_k^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T S_{k+1} \right)^{-1} \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} + Q_k. \tag{4.15}$$

Persamaan (4.15) merupakan persamaan Riccati. Persamaan Riccati tersebut akan ditentukan solusi S_k. Kemudian dari persamaan (4.7) dapat dibentuk persamaan sebagai berikut:

$$\lambda_{k+1} = \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T\right)^{-1} (\lambda_k - Q_k \boldsymbol{x}_k) \tag{4.16}$$

Berdasarkan persamaan (4.11) maka persamaan (4.16) menjadi:

$$\lambda_{k+1} = \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T\right)^{-1} (S_k \boldsymbol{x}_k - Q_k \boldsymbol{x}_k)$$

$$= \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T\right)^{-1} (S_k - Q_k)\boldsymbol{x}_k, \tag{4.17}$$

Kemudian substitusikan persamaan (4.17) ke persamaan (4.9) menjadi:

$$\boldsymbol{u}_k = -{R_k}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \lambda_{k+1}$$

$$= -{R_k}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T\right)^{-1} (S_k - Q_k)\boldsymbol{x}_k, \tag{4.18}$$

Selanjutnya persamaan (4.18) disubstitusikan ke persamaan (4.3) untuk mendapatkan sistem dinamik:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 \\ 1 \end{bmatrix} {R_k}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T\right)^{-1} (S_k - Q_k)\boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} {R_k}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T\right)^{-1} (S_k - Q_k)\right)\boldsymbol{x}_k \tag{4.19}$$

Persamaan (4.19) akan dianalisa menggunakan persamaan (2.8) dengan menentukan matriks S jika matriks S adalah definit positif maka persamaan (4.19) stabil.

Contoh 4.1:

Diberikan persamaan karakteristik sebagai berikut:

$$y(k) + 6y(k-1) + 8y(k-2) = u(k-1) + u(k-2)$$

Dengan fungsi tujuan memiliki matriks $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$,

Persoalan sistem dinamik untuk waktu diskrit dengan $k = 1, 2, ...13$. Kemudian diketahui $S_{13} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

Tentukan:

a. Bentuk kanonik terkontrol dari *Pulse Transfer*.

b. Tentukan fungsi kendali.

c. Analisa kestabilan.

Penyelesaian:

Diberikan persamaan karakteristik sebagai berikut:

$$y(k) + 6y(k-1) + 8y(k-2) = u(k-1) + u(k-2)$$

Dengan transformasi z sebagai berikut:

$y(k) = y(z)$ dan $y(k-i) = z^{-i}y(k)$ dan $u(k) = u(z)$ dan $u(k-i) = z^{-i}u(z)$

Maka persamaan karakteristik menjadi:

$$y(z) + 6z^{-1}y(z) + 8z^{-2}y(z) = z^{-1}u(z) + z^{-2}u(z)$$

$$y(z)(1 + a_1z^{-1} + a_2z^{-2}) = u(z)(z^{-1} + z^{-2})$$

$$y(z)(1 + 6z^{-1} + 8z^{-2}) = u(z)(z^{-1} + z^{-2})$$

$$\frac{y(z)}{u(z)} = \frac{z^{-1}+z^{-2}}{1+6z^{-1}+8z^{-2}}$$

atau

$$\frac{y(z)}{u(z)} = \frac{z+1}{z^2+6z+8}$$

maka diperoleh fungsi *Pulse Transfer* sebagai berikut:

$$\frac{y(z)}{u(z)} = \frac{z+1}{z^2+6z+8}$$

Oleh karena itu dapat dibentuk kanonik terkontrol sebagai berikut:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Maka didapat matriks A dan B sebagai berikut:

$A = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$ dan $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Dengan fungsi tujuan

$$J = \frac{1}{2}\boldsymbol{x}_N^T S_N \boldsymbol{x}_N + \frac{1}{2}\sum_{k=1}^{12}\left(\boldsymbol{x}_k^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k\right)$$

$$= \frac{1}{2}\boldsymbol{x}_N^T S_N \boldsymbol{x}_N + \frac{1}{2}\sum_{k=1}^{12}\left(\boldsymbol{x}_k^T \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\boldsymbol{x}_k + \boldsymbol{u}_k^T(1)\boldsymbol{u}_k\right)$$

Dari fungsi dinamik dengan fungsi tujuan dapat dibentuk persamaan Hamilton, persamaan *state*, persamaan *costate*, dan persamaan *stationer* sebagai berikut:

Persamaan Hamilton:

$$\boldsymbol{H} = \frac{1}{2}\left(\boldsymbol{x}^{\boldsymbol{T}}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\boldsymbol{x}_k + \boldsymbol{u}_k^T(1)\boldsymbol{u}_k\right) + \lambda_{k+1}^T\left(\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0\\1\end{bmatrix}\boldsymbol{u}_k\right)$$

Persamaan *state*: $\boldsymbol{x}_{k+1} = \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0\\1\end{bmatrix}\boldsymbol{u}_k$

Persamaan *costate*: $\lambda_k = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \lambda_{k+1}$

Persamaan *stationer*: $0 = \boldsymbol{u}_k + \begin{bmatrix}0\\1\end{bmatrix}^T \lambda_{k+1}$

Selanjutnya dari persamaan *stationer* diperoleh:

$$0 = \boldsymbol{u}_k + \begin{bmatrix}0\\1\end{bmatrix}^T \lambda_{k+1}$$

$$-\boldsymbol{u}_k = [0 \quad 1]\lambda_{k+1}$$

$$\boldsymbol{u}_k = -[0 \quad 1]\lambda_{k+1}$$

Kemudian disubstitusikan persamaan $\boldsymbol{u}_k$ ke persamaan *state* maka diperoleh:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0\\1\end{bmatrix}\boldsymbol{u}_k$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0\\1\end{bmatrix}(-[0 \quad 1]\lambda_{k+1})$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\lambda_{k+1}$$

Lalu dari persamaan *costate* diperoleh:

$$\lambda_k = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \lambda_{k+1}$$

Kemudian didefenisikan $\lambda_k = S_k\boldsymbol{x}_k$ sehingga diperoleh:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\lambda_{k+1}$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}S_{k+1}\boldsymbol{x}_{k+1}$$

$$\boldsymbol{x}_{k+1} = \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}S_{k+1}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k$$

Substitusikan $\lambda_k = S_k\boldsymbol{x}_k$ ke persamaan 4.10 sehingga diperoleh:

$$\lambda_k = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \lambda_{k+1}$$

$$S_k\boldsymbol{x}_k = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T S_{k+1}\boldsymbol{x}_{k+1}$$

Kemudian substitusikan $\boldsymbol{x}_{k+1}$ ke dalam $S_k\boldsymbol{x}_k$ sehingga diperoleh:

$$S_k\boldsymbol{x}_k = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T S_{k+1}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}S_{k+1}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}\boldsymbol{x}_k$$

$$S_k = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T S_{k+1}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}S_{k+1}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}$$

Maka diperoleh persamaan Riccati dengan solusi S_k. Selanjutnya akan dicari nilai dari S_k dengan cara menghitung mundur dari $k = 12$ sampai $k = 0$.

$$S_{12} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}$$
$$= \begin{bmatrix}-84,6 & -64,2\\-63,9 & -44,9\end{bmatrix}$$

$$S_{11} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \begin{bmatrix}-84,6 & -64,2\\-63,9 & -44,9\end{bmatrix}$$
$$\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-84,6 & -64,2\\-63,9 & -44,9\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}$$
$$= \begin{bmatrix}-61,6 & -35,0\\-35,8 & -12,8\end{bmatrix}$$

$$S_{10} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \begin{bmatrix}-61,6 & -35,0\\-35,8 & -12,8\end{bmatrix}$$
$$\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-61,6 & -35,0\\-35,8 & -12,8\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}$$
$$= \begin{bmatrix}-58,2 & -22,4\\-24,2 & -27,8\end{bmatrix}$$

$$S_9 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \begin{bmatrix}-58,2 & -22,4\\-24,2 & -27,8\end{bmatrix}$$
$$\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-58,2 & -22,4\\-24,2 & -27,8\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}$$
$$= \begin{bmatrix}-65,2 & -55,7\\-56,4 & -125,0\end{bmatrix}$$

$$S_8 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}^T \begin{bmatrix}-65,2 & -55,7\\-56,4 & -125,0\end{bmatrix}$$
$$\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-65,2 & -55,7\\-56,4 & -125,0\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 1\\-8 & -6\end{bmatrix}$$

$$= \begin{bmatrix} -62,5 & -44,6 \\ -44,1 & -70,1 \end{bmatrix}$$

$$S_7 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -62,5 & -44,6 \\ -44,1 & -70,1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -62,5 & -44,6 \\ -44,1 & -70,1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -62,3 & -42,5 \\ -42,4 & -61,8 \end{bmatrix}$$

$$S_6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -62,3 & -42,5 \\ -42,4 & -61,8 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -62,3 & -42,5 \\ -42,4 & -61,8 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -61,8 & -41,1 \\ -41,8 & -59,3 \end{bmatrix}$$

$$S_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -61,8 & -41,1 \\ -41,8 & -59,3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -61,8 & -41,1 \\ -41,8 & -59,3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -62,0 & -41,2 \\ -41,8 & -59,4 \end{bmatrix}$$

$$S_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -62,0 & -41,2 \\ -41,8 & -59,4 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -62,0 & -41,2 \\ -41,8 & -59,4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -62,0 & -42,3 \\ -41,8 & -59,4 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -62,0 & -42,3 \\ -41,8 & -59,4 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -62,0 & -42,3 \\ -41,8 & -59,4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -62,0 & -42,3 \\ -41,7 & -58,3 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -62,0 & -42,3 \\ -41,7 & -58,3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -62,0 & -42,3 \\ -41,7 & -58,3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -62,0 & -41,3 \\ -41,7 & -59,3 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -62,0 & -41,3 \\ -41,7 & -59,3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -62,0 & -41,3 \\ -41,7 & -59,3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -62,0 & -41,2 \\ -41,8 & -60,4 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}^T \begin{bmatrix} -62,0 & -41,3 \\ -41,7 & -59,3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -62,0 & -41,3 \\ -41,7 & -59,3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -62,0 & -41,2 \\ -41,8 & -60,4 \end{bmatrix}$$

Karena $S_1 = S_0$, maka hitung mundur berhenti dan didapat S_k adalah:

$$S_k = \begin{bmatrix} -62,0 & -41,2 \\ -41,8 & -60,4 \end{bmatrix}$$

Kemudian berdasarkan persamaan (4.18) maka didapat fungsi kendali yaitu:

$$\boldsymbol{u}_k = -R_k^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}^T \right)^{-1} (S_k - Q_k)\boldsymbol{x}_k,$$

$$= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} 0 & -8 \\ 1 & -6 \end{pmatrix}^{-1} \left(\begin{bmatrix} -62,0 & -41,2 \\ -41,8 & -60,4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \boldsymbol{x}_k$$

$$= \begin{bmatrix} -7,88 & -5,15 \end{bmatrix} \boldsymbol{x}_k$$

Maka diperoleh $\boldsymbol{u}_k = [-7,88 \quad -5,15]\boldsymbol{x}_k$, kemudian substitusikan kebentuk kanonik terkontrol sehingga didapat:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -7,88 & -5,15 \end{bmatrix} \boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \left(\begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -7,88 & -5,15 \end{bmatrix} \right) \boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ -15,9 & -11,2 \end{bmatrix} \boldsymbol{x}_k$$

Selanjutnya berdasarkan persamaan (2.8) maka dicari analisa kestabilanya sebagai berikut:

$$A_k^T S_k A_k - S_k = -Q_k,$$

$$\begin{bmatrix} 0 & -15,9 \\ 1 & -11,2 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -15,9 & -11,2 \end{bmatrix} - \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 253{,}0S_{22} - S_{11} & -16{,}9S_{12} + 178{,}0S_{22} \\ -16{,}9S_{12} + 178{,}0S_{22} & S_{11} - 22{,}4S_{12} + 124{,}0S_{22} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

maka diperoleh 3 persamaan sebagai berikut:

$$253{,}0S_{22} - S_{11} = -1$$

$$-16{,}9S_{12} + 178{,}0S_{22} = 0$$

$$S_{11} - 22{,}4S_{12} + 124{,}0S_{22} = -1$$

maka didapatkan $S_{11} = -1{,}53$, $S_{12} = -0{,}10$ dan $S_{22} = -0{,}01$, lalu buktikan dengan mencari nilai eigen:

$$S = \begin{bmatrix} -1{,}53 & -0{,}10 \\ -0{,}10 & -0{,}01 \end{bmatrix}$$

lalu dibentuk persamaan karakteristik, sebagai berikut:

$$\det(\lambda \mathrm{I} - S) = 0$$

$$det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1{,}53 & -0{,}10 \\ -0{,}10 & -0{,}01 \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} \lambda + 1{,}53 & 0{,}10 \\ 0{,}10 & \lambda + 0{,}01 \end{bmatrix}\right) = 0$$

$$\lambda^2 + 1{,}54\lambda = 0$$

$$\lambda_1 = -1{,}54 \text{ dan } \lambda_2 = 0$$

Berdasarkan perhitungan diatas diperoleh nilai eigennya $\lambda_1 = -1{,}54$ dan $\lambda_2 = 0$, maka persamanaan dinamik pada contoh diatas tidak stabil.

4.2 Sistem Dinamik Diskrit Untuk Matriks nxn

Diketahui persamaan karakteristik dinamik diskrit :

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n) \quad (4.20)$$

Dengan transformasi z sebagai berikut:

$y(k) = y(z)$ dan $y(k-i) = z^{-i} y(k)$ dan $u(k) = u(z)$ dan $u(k-i) = z^{-i} u(z)$

lalu diketahui persamaan *Pulse Transfer* (2.36) dari persamaan (2.36) yaitu:

$$\frac{y(z)}{u(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n} \quad (4.21)$$

Berdasarkan persamaan (4.21) direpresentasikan persamaan kanonik terkontrol waktu diskrit sebagai berikut:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k \tag{4.22}$$

Diberikan fungsi tujuan yaitu:

$$J = \frac{1}{2}\boldsymbol{x}_N^T S_N \boldsymbol{x}_N + \frac{1}{2}\sum_{k=1}^{N-1}(\boldsymbol{x}_k^T Q_k \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k), \tag{4.23}$$

Pada persamaan (4.23) diasumsikan Q_k, R_k, dan S_N adalah matrik simetri semi definit positif dan $R_k \neq 0$ untuk semua k.

Dari persamaan (4.22) dan (4.23) dibentuk persamaan Hamilton:

$$\boldsymbol{H} = \tfrac{1}{2}(\boldsymbol{x}^T Q_k \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k) + \lambda_{k+1}^T \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k \right), \tag{4.24}$$

Kemudian persamaan (4.5) menghasilkan persamaan *state*, *costate* dan *stasioner*

Persamaan *state* : $\boldsymbol{x}_{k+1} = \frac{\partial H}{\partial \lambda_{k+1}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k,$ (4.25)

Persamaan *costate* :

$$\lambda_k = \frac{\partial H}{\partial x_k} = Q_k \boldsymbol{x}_k + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}^T \lambda_{k+1}, \tag{4.26}$$

Persamaan *stationer* : $0 = \frac{\partial H}{\partial u_k} = R_k \boldsymbol{u}_k + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T \lambda_{k+1},$ (4.27)

Menurut persamaan (4.8) diperoleh:

$$\boldsymbol{u}_k = -(R_k)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T \lambda_{k+1}, \tag{4.28}$$

Kemudian persamaan (4.28) disubstitusikan ke persamaan (4.25):

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} {R_k}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T \lambda_{k+1}, \tag{4.29}$$

Menurut (**Ogata, 1995**) Selanjutnya diasumsikan untuk setiap k berlaku:

$$\lambda_k = S_k \boldsymbol{x}_k \tag{4.30}$$

Substitusikan persamaan (4.30) kedalam (4.29) untuk mendapatkan persamaan:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} {R_k}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T S_{k+1} \boldsymbol{x}_{k+1} \tag{4.31}$$

Pemecahan untuk $\boldsymbol{x}_{k+1}$ menghasilkan:

$$\boldsymbol{x}_{k+1} = \left(I + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} {R_k}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T S_{k+1} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \boldsymbol{x}_k \tag{4.32}$$

Selanjutnya, disubstitusikan persamaan (4.30) dengan persamaan *costate* (4.26):

$$S_k\boldsymbol{x}_k = Q_k\boldsymbol{x}_k + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}^T S_{k+1}\boldsymbol{x}_{k+1}, \qquad (4.33)$$

Kemudian disubstitusikan persamaan (4.32) dengan persamaan (4.33):

$$S_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}^T S_{k+1}\left(I + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} R_k^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T S_{k+1}\right)^{-1}$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} + Q_k, \qquad (4.34)$$

Persamaan (4.34) merupakan persamaan Riccati. Persamaan Riccati tersebut akan dicari solusi S_k. Kemudian dari persamaan (4.26) dapat dibentuk persamaan sebagai berikut:

$$\lambda_{k+1} = \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}^T\right)^{-1} (\lambda_k - Q_k\boldsymbol{x}_k), \quad (4.35)$$

Berdasarkan persamaan (4.30) maka persamaan (4.35) menjadi:

$$\lambda_{k+1} = \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}^T\right)^{-1} (S_k\boldsymbol{x}_k - Q_k\boldsymbol{x}_k)$$

$$= \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}^T\right)^{-1} (S_k - Q_k)\boldsymbol{x}_k, \quad (4.36)$$

Kemudian substitusikan persamaan (4.36) ke persamaan (4.28) menjadi:

$$\boldsymbol{u}_k = -R_k{}^{-1}\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}^T \lambda_{k+1}$$

$$= -R_k{}^{-1}\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}^T \left(\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}^T\right)^{-1} (S_k - Q_k)\boldsymbol{x}_k, \tag{4.37}$$

Selanjutnya persamaan (4.37) disubstitusikan ke persamaan (4.3) untuk mendapatkan sistem dinamik:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\boldsymbol{x}_k - \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} R_k{}^{-1}\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}^T$$

$$\left(\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}^T\right)^{-1} (S_k - Q_k)\boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \left(\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix} - \right.$$

$$\left.\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} R_k{}^{-1}\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}^T \left(\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}^T\right)^{-1} (S_k - Q_k)\right)\boldsymbol{x}_k \tag{4.38}$$

Persamaan (4.38) akan dianalisa menggunakan persamaan (2.8) dengan menentukan matriks S jika matriks S adalah definit positif maka persamaan (4.38) stabil.

Contoh 4.2:

Diberikan persamaan karakteristik sebagai berikut:

$$y(k) + 9y(k-1) + 24y(k-2) + 20y(k-3) = u(k-1) + 3u(k-2) + 2u(k-2)$$

Dengan fungsi tujuan memiliki matriks $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R = 1$,

Persoalan sistem dinamik untuk waktu diskrit dengan $k = 1$, 2,...13. Kemudian diketahui $S_{12} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 4 \end{bmatrix}$.

Tentukan:

d. Bentuk kanonik terkontrol dari *Pulse Transfer*.

e. Tentukan fungsi kendali.

f. Analisa kestabilan.

Penyelesaian:

Diberikan persamaan karakteristik sebagai berikut:

$$y(k) + 9y(k-1) + 24y(k-2) + 20y(k-3) = u(k-1) + 3u(k-2) + 2u(k-2)$$

Dengan transformasi z sebagai berikut:

$y(k) = y(z)$ dan $y(k-i) = z^{-i}y(k)$ dan $u(k) = u(z)$ dan $u(k-i) = z^{-i}u(z)$

Maka persamaan karakteristik menjadi:

$$y(z) + 9z^{-1}y(z) + 24z^{-2}y(z) + 20z^{-3}y(z) = z^{-1}u(z) + 3z^{-2}u(z) + 2z^{-2}u(z)$$

$$y(z)(1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3}) = u(z)(z^{-1} + 3z^{-2} + 2z^{-3})$$

$$y(z)(1 + 9z^{-1} + 24z^{-2} + 20z^{-3}) = u(z)(z^{-1} + 3z^{-2} + 2z^{-3})$$

$$\frac{y(z)}{u(z)} = \frac{z^{-1}+3z^{-2}+2z^{-3}}{1+9z^{-1}+24z^{-2}+20z^{-3}}$$

atau

$$\frac{y(z)}{u(z)} = \frac{z^2+3z+2}{z^3+9z^2+24z+20}$$

maka diperoleh fungsi *Pulse Transfer* sebagai berikut:

$$\frac{y(z)}{u(z)} = \frac{z^2 + 3z + 2}{z^3 + 9z^2 + 24z + 20}$$

Oleh karena itu dapat dibentuk kanonik terkontrol sebagai berikut:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Maka didapat matriks A dan B sebagai berikut:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \text{ dan } B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Dengan fungsi tujuan

$$\begin{aligned} J &= \frac{1}{2}\boldsymbol{x}_N^T S_N \boldsymbol{x}_N + \frac{1}{2}\sum_{k=1}^{12}\left(\boldsymbol{x}_k^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k\right) \\ &= \frac{1}{2}\boldsymbol{x}_N^T S_N \boldsymbol{x}_N + \frac{1}{2}\sum_{k=1}^{12}\left(\boldsymbol{x}_k^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{u}_k^T (1) \boldsymbol{u}_k\right) \end{aligned}$$

Dari fungsi dinamik dengan fungsi tujuan dapat dibentuk persamaan Hamilton, persamaan *state*, persamaan *costate*, dan persamaan *stationer* sebagai berikut:

Persamaan Hamilton:

$$\boldsymbol{H} = \frac{1}{2}\left(\boldsymbol{x^T} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{u}_k^T (1) \boldsymbol{u}_k\right) + \lambda_{k+1}^T \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k\right)$$

Persamaan *state*: $\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k$

Persamaan *costate*: $\lambda_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \lambda_{k+1}$

Persamaan *stationer*: $0 = \boldsymbol{u}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \lambda_{k+1}$

Selanjutnya dari persamaan *stationer* diperoleh:

$$0 = \boldsymbol{u}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \lambda_{k+1}$$

$$-\boldsymbol{u}_k = [0 \quad 0 \quad 1]\lambda_{k+1}$$

$$\boldsymbol{u}_k = -[0 \quad 0 \quad 1]\lambda_{k+1}$$

Kemudian disubstitusikan persamaan $\boldsymbol{u}_k$ ke persamaan *state* maka diperoleh:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (-[0 \quad 0 \quad 1]\lambda_{k+1})$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_{k+1}$$

Lalu dari persamaan *costate* diperoleh:

$$\lambda_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \lambda_{k+1}$$

Kemudian didefenisikan $\lambda_k = S_k \boldsymbol{x}_k$ sehingga diperoleh:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_{k+1}$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} S_{k+1} \boldsymbol{x}_{k+1}$$

$$\boldsymbol{x}_{k+1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} S_{k+1} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} \boldsymbol{x}_k$$

Substitusikan $\lambda_k = S_k \boldsymbol{x}_k$ ke persamaan 4.29 sehingga diperoleh:

$$\lambda_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \lambda_{k+1}$$

$$S_k \boldsymbol{x}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T S_{k+1} \boldsymbol{x}_{k+1}$$

Kemudian substitusikan $\boldsymbol{x}_{k+1}$ ke dalam $S_k \boldsymbol{x}_k$ sehingga diperoleh:

$$S_k\boldsymbol{x}_k = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\boldsymbol{x}_k +$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}^T S_{k+1}\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} -\right.$$

$$\left.\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}S_{k+1}\right)^{-1}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}\boldsymbol{x}_k$$

$$S_k = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} +$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}^T S_{k+1}\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} -\right.$$

$$\left.\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}S_{k+1}\right)^{-1}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}$$

Maka diperoleh persamaan Riccati dengan solusi S_k. Selanjutnya akan dicari nilai dari S_k dengan cara menghitung mundur dari $k = 12$ sampai $k = 0$.

$$S_{12} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}^T \begin{bmatrix}2 & 0 & 0\\0 & 0 & 3\\0 & 3 & 4\end{bmatrix}\left(\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} -\right.\right.$$

$$\left.\left.\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\0 & 0 & 3\\0 & 3 & 4\end{bmatrix}\right)\right)^{-1}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}$$

$$= \begin{bmatrix}-531{,}0 & -638{,}0 & -219{,}0\\-640{,}0 & 765{,}0 & -264{,}0\\-220{,}0 & -264{,}0 & -92{,}0\end{bmatrix}$$

$$S_{11} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} +$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}^T \begin{bmatrix}-531{,}0 & -638{,}0 & -219{,}0\\-640{,}0 & 765{,}0 & -264{,}0\\-220{,}0 & -264{,}0 & -92{,}0\end{bmatrix}\left(\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} -\right.\right.$$

$$\left.\left.\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-531{,}0 & -638{,}0 & -219{,}0\\-640{,}0 & 765{,}0 & -264{,}0\\-220{,}0 & -264{,}0 & -92{,}0\end{bmatrix}\right)\right)^{-1}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}$$

$$= \begin{bmatrix} -397,0 & -438,0 & -129,0 \\ -430,0 & -485,0 & -144,0 \\ -122,0 & -146,0 & -33,8 \end{bmatrix}$$

$$S_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -397,0 & -438,0 & -129,0 \\ -430,0 & -485,0 & -144,0 \\ -122,0 & -146,0 & -33,8 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -397,0 & -438,0 & -129,0 \\ -430,0 & -485,0 & -144,0 \\ -122,0 & -146,0 & -33,8 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -387,0 & -396,0 & -95,0 \\ -392,0 & -329,0 & -24,0 \\ -91,8 & -2,0 & -113,0 \end{bmatrix}$$

$$S_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -387,0 & -396,0 & -95,0 \\ -392,0 & -329,0 & -24,0 \\ -91,8 & -2,0 & -113,0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -387,0 & -396,0 & -95,0 \\ -392,0 & -329,0 & -24,0 \\ -91,8 & -2,0 & -113,0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -403,0 & -495,0 & -182,0 \\ -500,0 & -1090,0 & -623,0 \\ -178,0 & -596,0 & -408,0 \end{bmatrix}$$

$$S_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -403,0 & -495,0 & -182,0 \\ -500,0 & -1090,0 & -623,0 \\ -178,0 & -596,0 & -408,0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -403,0 & -495,0 & -182,0 \\ -500,0 & -1090,0 & -623,0 \\ -178,0 & -596,0 & -408,0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -397,0 & -468,0 & -179,0 \\ -468,0 & -871,0 & -411,0 \\ -149,0 & -409,0 & -246,0 \end{bmatrix}$$

$$S_7 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} +$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}^T \begin{bmatrix}-397,0 & -468,0 & -179,0\\-468,0 & -871,0 & -411,0\\-149,0 & -409,0 & -246,0\end{bmatrix} \left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} -\right.$$

$$\left.\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}-397,0 & -468,0 & -179,0\\-468,0 & -871,0 & -411,0\\-149,0 & -409,0 & -246,0\end{bmatrix}\right)^{-1} \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}$$

$$= \begin{bmatrix}-397,0 & -468,0 & -169,0\\-464,0 & -826,0 & -359,0\\-146,0 & -393,0 & -245,0\end{bmatrix}$$

$$S_6 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} +$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}^T \begin{bmatrix}-397,0 & -468,0 & -169,0\\-464,0 & -826,0 & -359,0\\-146,0 & -393,0 & -245,0\end{bmatrix} \left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} -\right.$$

$$\left.\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}-397,0 & -468,0 & -169,0\\-464,0 & -826,0 & -359,0\\-146,0 & -393,0 & -245,0\end{bmatrix}\right)^{-1} \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}$$

$$= \begin{bmatrix}-397,0 & -468,0 & -159,0\\-464,0 & -846,0 & -389,0\\-149,0 & -423,0 & -296,0\end{bmatrix}$$

$$S_5 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} +$$

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}^T \begin{bmatrix}-397,0 & -468,0 & -159,0\\-464,0 & -846,0 & -389,0\\-149,0 & -423,0 & -296,0\end{bmatrix} \left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} -\right.$$

$$\left.\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}-397,0 & -468,0 & -159,0\\-464,0 & -846,0 & -389,0\\-149,0 & -423,0 & -296,0\end{bmatrix}\right)^{-1} \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-20 & -24 & -9\end{bmatrix}$$

$$= \begin{bmatrix}-399,0 & -470,0 & -130,0\\-468,0 & -861,0 & -331,0\\-153,0 & -448,0 & -328,0\end{bmatrix}$$

$$S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -399{,}0 & -470{,}0 & -130{,}0 \\ -468{,}0 & -861{,}0 & -331{,}0 \\ -153{,}0 & -448{,}0 & -328{,}0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -399{,}0 & -470{,}0 & -130{,}0 \\ -468{,}0 & -861{,}0 & -331{,}0 \\ -153{,}0 & -448{,}0 & -328{,}0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -397{,}0 & -468{,}0 & -139{,}0 \\ -470{,}0 & -893{,}0 & -412{,}0 \\ -159{,}0 & -499{,}0 & -461{,}0 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -397{,}0 & -468{,}0 & -139{,}0 \\ -470{,}0 & -893{,}0 & -412{,}0 \\ -159{,}0 & -499{,}0 & -461{,}0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -397{,}0 & -468{,}0 & -139{,}0 \\ -470{,}0 & -893{,}0 & -412{,}0 \\ -159{,}0 & -499{,}0 & -461{,}0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -397{,}0 & -468{,}0 & -159{,}0 \\ -476{,}0 & -930{,}0 & -614{,}0 \\ -162{,}0 & -524{,}0 & -512{,}0 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -397{,}0 & -468{,}0 & -159{,}0 \\ -476{,}0 & -930{,}0 & -614{,}0 \\ -162{,}0 & -524{,}0 & -512{,}0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -397{,}0 & -468{,}0 & -159{,}0 \\ -476{,}0 & -930{,}0 & -614{,}0 \\ -162{,}0 & -524{,}0 & -512{,}0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -397{,}0 & -458{,}0 & -79{,}0 \\ -472{,}0 & -895{,}0 & -412{,}0 \\ -156{,}0 & -463{,}0 & -359{,}0 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -397,0 & -458,0 & -79,0 \\ -472,0 & -895,0 & -412,0 \\ -156,0 & -463,0 & -359,0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -397,0 & -458,0 & -79,0 \\ -472,0 & -895,0 & -412,0 \\ -156,0 & -463,0 & -359,0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -399,0 & -470,0 & -180,0 \\ -474,0 & -928,0 & -613,0 \\ -157,0 & -480,0 & -430,0 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \begin{bmatrix} -399,0 & -470,0 & -180,0 \\ -474,0 & -928,0 & -613,0 \\ -157,0 & -480,0 & -430,0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -399,0 & -470,0 & -180,0 \\ -474,0 & -928,0 & -613,0 \\ -157,0 & -480,0 & -430,0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -399,0 & -470,0 & -180,0 \\ -474,0 & -928,0 & -613,0 \\ -157,0 & -480,0 & -430,0 \end{bmatrix}$$

Karena $S_1 = S_0$, maka hitung mundur berhenti dan didapat S_k adalah:

$$S_k = \begin{bmatrix} -399,0 & -470,0 & -180,0 \\ -474,0 & -928,0 & -613,0 \\ -157,0 & -480,0 & -430,0 \end{bmatrix}$$

Kemudian berdasarkan persamaan (4.37) maka didapat fungsi kendali yaitu:

$$\boldsymbol{u}_k = -{R_k}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^T \right)^{-1} (S_k - Q_k)\boldsymbol{x}_k,$$

$$= -\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}^{-1} \left(\begin{bmatrix} -399,0 & -470,0 & -180,0 \\ -474,0 & -928,0 & -613,0 \\ -157,0 & -480,0 & -430,0 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \boldsymbol{x}_k$$

$$= \begin{bmatrix} -20,0 & -23,5 & -9,00 \end{bmatrix} \boldsymbol{x}_k$$

Maka diperoleh $\boldsymbol{u}_k = [-20{,}0 \quad -23{,}5 \quad -9{,}00]\boldsymbol{x}_k$, kemudian substitusikan kebentuk kanonik terkontrol sehingga didapat:

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix}\boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[-20{,}0 \quad -23{,}5 \quad -9{,}00]\boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -20{,}0 & -23{,}5 & -9{,}00 \end{bmatrix}\right)\boldsymbol{x}_k$$

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -40 & -47{,}5 & -18 \end{bmatrix}\boldsymbol{x}_k$$

Selanjutnya berdasarkan persamaan (2.8) maka dicari analisa kestabilanya sebagai berikut:

$$A_k^T S_k A_k - S_k = -Q_k,$$

$$\begin{bmatrix} 0 & 0 & -40 \\ 1 & 0 & -47{,}5 \\ 0 & 1 & -18 \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -40 & -47{,}5 & -18 \end{bmatrix} - \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1600S_{33} - S_{11} & -40S_{31} + 1900S_{33} - S_{12} & -40S_{32} + 720S_{33} - S_{13} \\ -40S_{13} + 1900S_{33} - S_{21} & S_{11} - 47{,}5S_{31} - 47{,}5S_{13} + 2260S_{33} - S_{22} & S_{12} - 47{,}5S_{32} - 18S_{13} + 855S_{33} - S_{23} \\ -40S_{23} + 720S_{33} - S_{31} & S_{21} - 18S_{31} - 47{,}5S_{23} + 855S_{33} - S_{32} & S_{22} - 18S_{32} - 18S_{23} + 323S_{33} \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

maka diperoleh persamaan sebagai berikut:

$$1600S_{33} - S_{11} = -1$$

$$-40S_{31} + 1900S_{33} - S_{12} = 0$$

$$-40S_{32} + 720S_{33} - S_{13} = 0$$

$$-40S_{13} + 1900S_{33} - S_{21} = 0$$

$$S_{11} - 47{,}5S_{31} - 47{,}5S_{13} + 2260S_{33} - S_{22} = -1$$

$$S_{12} - 47{,}5S_{32} - 18S_{13} + 855S_{33} - S_{23} = 0$$

$$-40S_{23} + 720S_{33} - S_{31} = 0$$

$$S_{21} - 18S_{31} - 47{,}5S_{23} + 855S_{33} - S_{32} = 0$$

$$S_{22} - 18S_{32} - 18S_{23} + 323S_{33} = -1$$

maka didapatkan $S_{11} = -10{,}52$, $S_{12} = -4{,}14$, $S_{13} = -0{,}24$, $S_{21} = -4{,}14$, $S_{22} = -3{,}13$, $S_{23} = -0{,}12$, $S_{31} = -0{,}24$, $S_{32} = -0{,}12$, $S_{33} = -0{,}01$, lalu buktikan dengan mencari nilai eigen:

$$S = \begin{bmatrix} -10{,}52 & -4{,}14 & -0{,}24 \\ -4{,}14 & -3{,}13 & -0{,}12 \\ -0{,}24 & -0{,}12 & -0{,}01 \end{bmatrix}$$

lalu dibentuk persamaan karakteristik, sebagai berikut:

$$\det(\lambda \mathrm{I} - S) = 0$$

$$det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -10{,}52 & -4{,}14 & -0{,}24 \\ -4{,}14 & -3{,}13 & -0{,}12 \\ -0{,}24 & -0{,}12 & -0{,}01 \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} \lambda + 10{,}52 & 4{,}14 & 0{,}24 \\ 4{,}14 & \lambda + 3{,}13 & 0{,}12 \\ 0{,}24 & 0{,}12 & \lambda + 0{,}01 \end{bmatrix}\right) = 0$$

$$\lambda^3 + 13{,}66\lambda^2 + 15{,}8525\lambda + 0{,}064568 = 0$$

$$\lambda_1 = -12{,}38,\ \lambda_2 = -1{,}27, \text{ dan } \lambda_3 = -0{,}004$$

Berdasarkan perhitungan diatas diperoleh nilai eigennya $\lambda_1 = -12{,}38$, $\lambda_2 = -1{,}27$, dan $\lambda_3 = -0{,}004$, maka persamanaan dinamik pada contoh diatas tidak stabil.